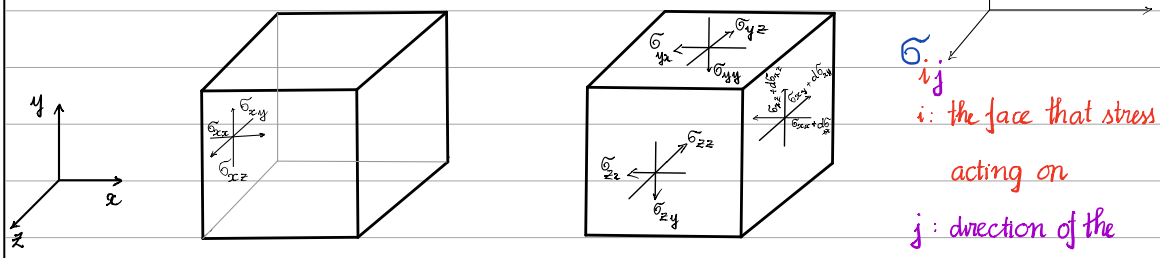
Strong form of 3D case:



i: the face that stress is acting on

j: direction of the applied stress

- Equilibrium in x direction:

$$(\sigma_{xx}+d\sigma_{xx})dydz - \sigma_{xx}dydz + (\sigma_{xy}+d\sigma_{xy})d_xd_z - \sigma_{xy}d_xd_z + (\sigma_{xz}+d\sigma_{xz})d_xd_y - \sigma_{xz}d_xd_y + \underbrace{b_x d_x d_y d_z}_{\text{external force}} = 0$$

$$\Leftrightarrow \quad d\sigma_{xx}dydz + d\sigma_{xy}dxdz + d\sigma_{xz}dxdy + b_x d_x d_y d_z = 0 \qquad (1)$$

Notice that:

$$d\sigma_{xx} = \frac{\partial \sigma_{xx}}{\partial x}dx \;;\; d\sigma_{xy} = \frac{\partial \sigma_{xy}}{\partial y}dy \;;\; d\sigma_{xz} = \frac{\partial \sigma_{xz}}{\partial z}dz$$

$(1) \Leftrightarrow$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x = 0$$

Similarly, in y and z direction:

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + b_y = 0$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = 0$$

$$\Rightarrow \begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} + \dfrac{\partial \sigma_{xz}}{\partial z} + b_x = 0 \\ \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \sigma_{yz}}{\partial z} + b_y = 0 \\ \dfrac{\partial \sigma_{xz}}{\partial x} + \dfrac{\partial \sigma_{yz}}{\partial y} + \dfrac{\partial \sigma_{zz}}{\partial z} + b_z = 0 \end{cases}$$

Notice that:
$\sigma_{xy} = \sigma_{yx}$
$\sigma_{yz} = \sigma_{zx}$
$\sigma_{xz} = \sigma_{zx}$

$\Rightarrow$ Static equilibrium equation in 3D:

$$\nabla_S^T \sigma + b = 0$$

in which:

$$b = \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix}$$: External force; body forces acting per unit volume

$$\nabla_S = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ \partial/\partial y & \partial/\partial x & 0 \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \end{bmatrix} ; \quad \sigma = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{bmatrix}$$

Topic: _____ Notebook