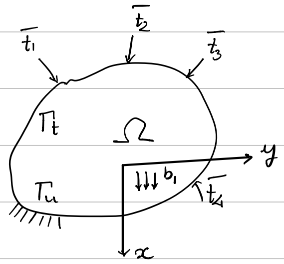
Topic: 3D Strong Form to Weak Form **Notebook**

① Strong form:

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} + \dfrac{\partial \sigma_{xz}}{\partial z} + b_x = 0 & (1) \\ \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \sigma_{yz}}{\partial z} + b_y = 0 & (2) \\ \dfrac{\partial \sigma_{xz}}{\partial x} + \dfrac{\partial \sigma_{yz}}{\partial y} + \dfrac{\partial \sigma_{zz}}{\partial z} + b_z = 0 & (3) \end{cases}$$



- Displacement conditions: (Dirichlet condition)

$$u_x = \bar{u}_x \text{ on } \Gamma_{ux}; \quad u_y = \bar{u}_y \text{ on } \Gamma_{uy}; \quad u_z = \bar{u}_z \text{ on } \Gamma_{uz}$$

- Force boundary condition (Neumann condition)

$$\vec{\sigma}_x \vec{n} = t_x \text{ on } \Gamma_{tx}; \quad \vec{\sigma}_y \vec{n} = t_y \text{ on } \Gamma_{ty}; \quad \vec{\sigma}_z \vec{n} = t_z$$

where:

$$\sigma_x = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{xz} \end{Bmatrix}; \quad \sigma_y = \begin{Bmatrix} \sigma_{yx} \\ \sigma_{yy} \\ \sigma_{yz} \end{Bmatrix}; \quad \sigma_z = \begin{Bmatrix} \sigma_{zx} \\ \sigma_{zy} \\ \sigma_{zz} \end{Bmatrix}$$

- Notice:

$$\Gamma_{ux} \cup \Gamma_{tx} = \Gamma_{uy} \cup \Gamma_{ty} = \Gamma; \quad \Gamma_{ux} \cap \Gamma_{tx} = \Gamma_{uy} \cap \Gamma_{ty} = \phi$$

② Weak form:

- Consider an arbitrary vector field $\vec{v}(x,y,z) = \begin{cases} v_x(x,y,z) \\ v_y(x,y,z) \\ v_z(x,y,z) \end{cases}$, whose components vanish at the corresponding bounding segments: $v = 0$ on Γ_u

- Multiplying (1) & (2), (3) by v_x. Then integrate the resulting expression over Ω.

$$\int_\Omega v_x \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x \right) dV = 0$$

- Integration by part: $\int_\Omega (ab)' d\Omega = \int_\Omega a'b \, d\Omega + \int_\Omega b'a \, d\Omega$

$$\Rightarrow \int_\Omega v_x \frac{\partial \sigma_{xx}}{\partial x} d\Omega = -\int_\Omega \frac{\partial v_x}{\partial x} \sigma_{xx} \, d\Omega + \int_\Omega \frac{\partial}{\partial x} (v_x \sigma_{xx}) d\Omega$$

$$\int_\Omega v_x \frac{\partial \sigma_{xy}}{\partial x} d\Omega = -\int_\Omega \frac{\partial v_x}{\partial x} \sigma_{xy} \, d\Omega + \int_\Omega \frac{\partial}{\partial x} (v_x \sigma_{xy}) d\Omega$$

$$\int_\Omega v_x \frac{\partial \sigma_{xz}}{\partial x} d\Omega = -\int_\Omega \frac{\partial v_x}{\partial xz} \sigma_{xz} \, d\Omega + \int_\Omega \frac{\partial}{\partial x} (v_x \sigma_{xz}) d\Omega$$

Reagranging:

$$\iiint_\Omega \Big[\Big\{ \frac{\partial v_x}{\partial x}\sigma_{xx} + \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\sigma_{xy}$$

$$+ \frac{\partial v_y}{\partial y}\sigma_{yy} + \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)\sigma_{yz}$$

$$+ \frac{\partial v_z}{\partial z}\sigma_{zz} + \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)\sigma_{zx} \Big\}$$

$$+ \left\{ u_x b_x + u_y b_y + u_z b_z \right\}$$

$$+ \Big\{ \frac{\partial}{\partial x}(v_x \sigma_{xx}) + \frac{\partial}{\partial y}(v_x \sigma_{xy}) + \frac{\partial}{\partial z}(v_x \sigma_{xz})$$

$$+ \frac{\partial}{\partial x}(v_y \sigma_{yx}) + \frac{\partial}{\partial y}(v_y \sigma_{yy}) + \frac{\partial}{\partial z}(v_y \sigma_{yz})$$

$$+ \frac{\partial}{\partial x}(v_z \sigma_{zx}) + \frac{\partial}{\partial y}(v_z \sigma_{zy}) + \frac{\partial}{\partial z}(v_z \sigma_{zz}) \Big\} \Big] dV = 0$$

- Divergence theorem:

$$\oint_\Omega \nabla F \, d\Omega = \oint_S n F \, ds$$

$\Rightarrow$ Blue

$$= \int_\Gamma \Big\{ v_x (n_x \sigma_{xx} + n_y \sigma_{xy} + n_z \sigma_{xz})$$

$$+ v_y (n_x \sigma_{yx} + n_y \sigma_{yy} + n_z \sigma_{yz})$$

$$+ v_z (n_x \sigma_{zx} + n_y \sigma_{zy} + n_z \sigma_{zz}) \Big\} dS$$

$$= \int_\Gamma (v_x t_x + v_y t_y + v_z t_z)\, dS \qquad \boxed{\begin{cases} t_x = n_x \sigma_{xx} + n_y \sigma_{xy} + n_z \sigma_{xz} \\ t_y = n_x \sigma_{yx} + n_y \sigma_{yy} + n_z \sigma_{yz} \\ t_z = n_x \sigma_{zx} + n_y \sigma_{zy} + n_z \sigma_{zz} \end{cases}}$$

$$= \int_\Gamma v^T t \, dS$$

$$= \underbrace{\cancel{\int_{\Gamma_u} v^T t \, dS}}_{v = 0 \text{ on } \Gamma_u} + \int_{\Gamma_t} v^T t \, dS = \int_{\Gamma_t} v^T t \, dS$$

Tractions acting per unit area of external boundary surface Γ of the solid

$$\nabla_s = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ \partial/\partial y & \partial/\partial x & 0 \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial z \end{bmatrix} ; \quad \sigma = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{bmatrix} ; \quad v = \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix}$$

$$\underset{(6\times3)\times(3\times1)}{\nabla_s v} = \begin{bmatrix} \partial v_x/\partial x \\ \partial v_y/\partial y \\ \partial v_z/\partial z \\ \partial v_x/\partial y + \partial v_y/\partial x \\ \partial v_y/\partial z + \partial v_z/\partial y \\ \partial v_x/\partial z + \partial v_z/\partial x \end{bmatrix}$$

$$\Rightarrow (\nabla_s v)^T \sigma = \frac{\partial v_x}{\partial x}\sigma_x + \frac{\partial v_y}{\partial y}\sigma_y + \frac{\partial v_z}{\partial z}\sigma_z + \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\sigma_{xy} + \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)\sigma_{yz} + \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)\sigma_{xz}$$

WEAK CONTINUOUS FORM

$$\Rightarrow \int_\Omega (\nabla_s v)^T \sigma \, dV = \int_\Omega v^T b \, dV + \int_{\Gamma_t} v^T t \, dS$$

$\Rightarrow$ Strong form with 2^{nd} derivative of displacement has been changed to weak form with 1^{st} derivative of displacement