1 - Strain - displacement matrix of element:

Displacement components of element u^e are interpolated from the node displacement d^e through shape function matrix of elements $N^e(x)$:

$$u^e = N^e(x)\, d^e$$

in which:

$$u^e = \begin{bmatrix} u_1^e(x) \\ u_2^e(x) \\ \vdots \\ u_{n_d}^e(x) \end{bmatrix}$$

Displacement vector d^e of element Ω^e

$$d^e = \begin{bmatrix} d_1^e \\ d_2^e \\ \vdots \\ d_{n_d}^e \end{bmatrix} = \begin{bmatrix} \left.\begin{matrix} d_{11}^e \\ d_{12}^e \\ \vdots \\ d_{1n_n}^e \end{matrix}\right\} d_1^e \\ \left.\begin{matrix} d_{21}^e \\ d_{22}^e \\ \vdots \\ d_{2n_n}^e \end{matrix}\right\} d_2^e \\ \left.\begin{matrix} d_{n_d1}^e \\ d_{n_d2}^e \\ \vdots \\ d_{n_dn_n}^e \end{matrix}\right\} d_{n_d}^e \end{bmatrix}$$

In the real calculation of FEM, we usually arrange the displacement vector d^e of element Ω^e in the nodal order:

$$d^e = \begin{bmatrix} \left.\begin{matrix} d_{11}^e \\ d_{21}^e \\ \vdots \\ d_{n_d1}^e \end{matrix}\right\} n_d \text{ displacement of node } 1 \\ \left.\begin{matrix} d_{12}^e \\ d_{22}^e \\ \vdots \\ d_{n_d2}^e \end{matrix}\right\} n_d \text{ displacement of node } 2 \\ \left.\begin{matrix} d_{1n_n}^e \\ d_{2n_n}^e \\ \vdots \\ d_{n_dn_n}^e \end{matrix}\right\} n_d \text{ displacement of node } n_n \end{bmatrix}$$

Shape function $N^e(x)$ matrix of element Ω^e is described:

$$\mathbb{N}^e = \begin{bmatrix} N_1^e(x) & 0 & 0 & \dots & 0 & N_2^e(x) & 0 & 0 & \dots & 0 & \dots & N_{n_n}^e(x) & 0 & 0 & \dots & 0 \\ 0 & N_1^e(x) & 0 & \dots & 0 & 0 & N_2^e(x) & 0 & \dots & 0 & \dots & 0 & N_{n_n}^e(x) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ \underbrace{0 \quad 0 \quad 0 \quad \dots \quad N_1^e(x)}_{1^{st}\text{ node, } n_d \text{ components}} & & & & & \underbrace{0 \quad 0 \quad 0 \quad \dots \quad N_2^e(x)}_{2^{nd}\text{ node, } n_d \text{ components}} & & & & & & \underbrace{0 \quad 0 \quad 0 \quad \dots \quad N_{n_n}^e(x)}_{n_n^{th}\text{ node, } n_d \text{ components}} \end{bmatrix}$$

Or in concise form:

$$N^e(x) = [N_1^e(x) \quad N_2^e(x) \dots N_{n_n}^e(x)]$$

in which $N_I^e(x)$, $I = 1, 2, \dots n_x$ is shape function matrix of element Ω^e corresponding to node I

$$N_I^e(x) = \begin{bmatrix} N_I^e(x) & 0 & \dots & 0 \\ 0 & N_I^e(x) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & N_I^e(x) \end{bmatrix}$$

Strain-displacement matrix of element:

ε_{xx}
ε_{yy}
ε_{zz}
ε_{xy}
ε_{xz}
ε_{zx}

$$\varepsilon^e = \frac{du^e}{dx} = \frac{\partial N^e(x)}{\partial x} d^e = B^e(x) d^e$$

$$\mathbb{B}^e(x) = \nabla_S \mathbb{N}^e(x) = [\nabla_S N_1^e(x) \quad \nabla_S N_2^e(x) \cdots \nabla_S N_{n_n}^e(x)]$$

$$= [\mathbb{B}_1^e(x) \quad \mathbb{B}_2^e(x) \dots \mathbb{B}_{n_n}^e(x)]$$

$\Rightarrow$ in which $\mathbb{B}^e(x)$ is strain-displacement matrix of element corresponding to node I

2 Derivation of system equations:

From the continuous weak form, we will change into a discrete one. In other words, instead of finding an unknown function, we want to find "n" unknowns. We will need a system of discrete equations, and eventually obtain the system equation in the form

$$\mathbb{K}\mathbb{U} = \mathbb{F}$$

K is the stiffness of the system, U is the displacement vector of nodes. $\mathbb{F}$ is the vector of forces applied to the systems. The following will the following in detail

Interpolation of displacements by using shape function $N(x)$ and nodal displacement, d:

$$u = N(x)d = [N_1(x)\ N_2(x) \dots N_{n_n}(x)] \begin{Bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n_n} \end{Bmatrix} \quad (*)$$

Weak form:

$$\int_\Omega (\nabla_s v)^T \sigma \, d\Omega = \int_\Omega v^T b \, d\Omega + \int_{\Gamma_t} v^T t \, d\Gamma_t$$

From continuous weak form, we choose N_n test function $v_1(x)$, $v_2(x), \dots, v_{n_n}(x)$. Each function gives one equation. In Galerkin FEM, method we simply choose the test functions $v_1(x), v_2(x), v_3(x)$ the same as shape functions $N_1(x), N_2(x), \dots, N_{n_n}(x)$. Substituting these N_n functions of $v(x)$ into $(*)$.

$$\int_\Omega (\nabla_s N_I)^T (D \nabla_s u) \, d\Omega = \int_\Omega N_I^T b \, d\Omega + \int_{\Gamma_t} (N_I^T \cdot \bar{t}) \, d\Gamma$$

in which $I = 1, 2, \dots, N_n$.

- Using $B(x) = \nabla_s N(x)$ and $\varepsilon = \frac{du}{dx} = \frac{\partial N(x)}{dx} d = B(x) d$

$$\Rightarrow \left[\int_\Omega B_I^T D B \, d\Omega \right] d = \int_\Omega (N_I^T b) \, d\Omega + \int_{\Gamma_t} (N_I^T \bar{t}) \, d\Gamma, \quad I = 1, 2, \dots, N_n \quad (3)$$

in which the transpose of the global strain-displacement matrix is:

$$B_I^T = \begin{bmatrix} B_1^T \\ B_2^T \\ \vdots \\ B_{n_n}^T \end{bmatrix} \rightarrow \begin{bmatrix} B_{1x} \\ B_{1y} \\ B_{1z} \\ B_{2x} \\ B_{2y} \\ B_{2z} \\ \vdots \end{bmatrix}$$

$(3) \Rightarrow$

$$\begin{cases} \int_\Omega \left(B_1^T D [B_1\ B_2 \dots B_{n_n}] \, d\Omega \right) d = \int_\Omega N_1^T b \, d\Omega + \int_{\Gamma_t} N_1^T t \, d\Gamma \\ \int_\Omega \left(B_2^T D [B_1\ B_2 \dots B_{n_n}] \, d\Omega \right) d = \int_\Omega N_2^T b \, d\Omega + \int_{\Gamma_t} N_2^T t \, d\Gamma \\ \vdots \\ \int_\Omega \left(B_{n_n}^T D [B_1\ B_2 \dots B_{n_n}] \, d\Omega \right) d = \int_\Omega N_{n_n}^T b \, d\Omega + \int_{\Gamma_t} N_{n_n}^T t \, d\Gamma \end{cases}$$

$\mathbb{B}_1 \to$ 1D: 1 component
2D: 2 component
3D: 3 component

Or we can write in the matrix form:

$$\underbrace{\begin{bmatrix} \int_\Omega B_1^T D B_1 d\Omega & \int_\Omega B_1^T D B_2 d\Omega & \dots & \int_\Omega B_1^T D B_{n_n} d\Omega \\ \int_\Omega B_2^T D B_1 d\Omega & \int_\Omega B_2^T D B_2 d\Omega & \dots & \int_\Omega B_2^T D B_{n_n} d\Omega \\ \vdots & \vdots & & \vdots \\ \int_\Omega B_{n_n}^T D B_1 d\Omega & \int_\Omega B_{n_n}^T D B_2 d\Omega & \dots & \int_\Omega B_{n_n}^T D B_{n_n} d\Omega \end{bmatrix}}_{\text{Global stiffness matrix } \mathbb{K}} \underbrace{\begin{Bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n_n} \end{Bmatrix}}_{\text{Displacement}} = \underbrace{\begin{bmatrix} \int_\Omega N_1^T b \, d\Omega + \int N_1^T t \, d\Gamma \\ \int_\Omega N_2^T b \, d\Omega + \int N_2^T t \, d\Gamma \\ \vdots \\ \int_\Omega N_{n_n}^T b \, d\Omega + \int N_{n_n}^T t \, d\Gamma \end{bmatrix}}_{\text{Force vector}}$$

$$\mathbb{K} = \begin{bmatrix} \mathbb{K}_{11} & \mathbb{K}_{12} & \dots & \mathbb{K}_{1n_n} \\ \mathbb{K}_{21} & \mathbb{K}_{22} & \dots & \mathbb{K}_{2n_n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{K}_{n_n 1} & \mathbb{K}_{n_n 2} & \dots & \mathbb{K}_{n_n n_n} \end{bmatrix}$$

Stiffness matrix:

In FEM, we firstly calculate stiffness matrix components K_{IJ}, $I, J = 1, 2, \dots N_n$ based on Ω_e elements. Then we assemble them together:

$$K_{IJ} = \int_\Omega B_I^T D B_J \, d\Omega = \mathop{\mathbb{A}}_{e=1}^{N_e} \underbrace{\int_{\Omega^e} B_I^T D B_J \, d\Omega}_{\mathbb{K}^e_{IJ}} , \quad I, J = 1, 2, \dots, N_n$$

Global

$\mathbb{K}^e_{IJ} \to$ Gauss Integration

in which $\mathbb{A}$ denotes an assembly process of stiffness elements $\mathbb{K}^e_{IJ}$ to obtain $\mathbb{K}$ in the domain Ω.

⇒ Later will be explain.

- Remind that the calculation of K^e_{IJ} only base on Ω_e element. Therefore, we only consider the part inside Ω_e element and ignore the other

$$K^e_{IJ} = \int_{\Omega^e} (\mathbb{B}^e_I)^T D \mathbb{B}^e_J \, d\Omega , \quad I, J = 1, 2, \dots N_n$$

in which $\mathbb{B}^e$ is the parts of $\mathbb{B}$ matrix that are inside Ω^e element:
$\mathbb{B}^e_I = \nabla_S N^e_I$, $I = 1, 2, \dots N_n$

Topic:

Notebook

$$\left[\mathop{\mathbb{A}}_{e=1}^{N_e} \int_{\Omega^e} (\mathbb{B}_I^e)^T \mathbb{D} \mathbb{B}_J^e \, d\Omega \right] \mathbb{d} = \mathop{\mathbb{A}}_{e=1}^{N_e} \int \mathbb{N}_J^T(x) \mathbb{b} \, d\Omega + \int_{\Gamma_t^e} \mathbb{N}_J^T(x) \mathbb{t} \, d\Gamma$$