FDM - 1D

Solve : $-\frac{du^2}{dx^2} = 1 \quad ; [0,1]$ → $Ku = F$

BC : $u(0) = 0 \; ; \; u(1) = 0$

Exact solution : $u = -\frac{1}{2}x^2 + \frac{1}{2}$

* First differences:

Forward : $\Delta_F u = \frac{u(x+h) - u(x)}{h}$

Backward : $\Delta_B u = \frac{u(x) - u(x-h)}{h}$

Centered : $\Delta_C u = \frac{u(x+h) - u(x-h)}{2h}$

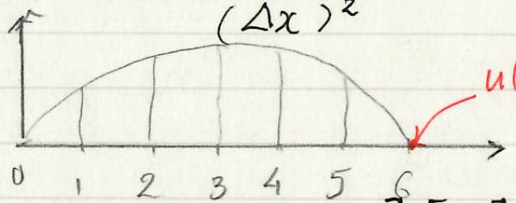
$$u(x+h) = u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''$$

$$u(x-h) = u(x) - hu'(x) + \frac{h^2}{2}u''(x) - \frac{h^3}{6}u'''$$

$$\Rightarrow u(x+h) + u(x-h) = 2u(x) + h^2 u''(x)$$

$$\Rightarrow u'' = \frac{-u_{i+1} + 2u_i - u_{i-1}}{(\Delta x)^2} \qquad (i = 1,5)$$

(1)



$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (KU = F)$$

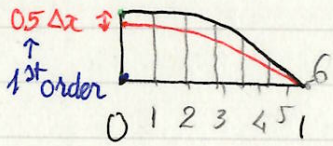
- $i = 6 \Rightarrow \frac{1}{\Delta x^2}(-u_6 + 2u_5 - u_4) = 1$ ($u_6 \to 0$)
- $i = 1 \Rightarrow \frac{1}{\Delta x^2}(-u_2 + 2u_1 - u_0) = 0$ ($u_0 \to U$)
- $i = 0 \Rightarrow \frac{1}{\Delta x^2}(-u_1 + 2u_0 - u_{-1}) = 0$

FDM - 1D

Solve : $-\frac{du^2}{dx^2} = 1 \quad ; [0,1]$

BC : $u(1) = 0 \quad ; du/dx\,(0) = 0$

Exact solution : $u(x) = -\frac{1}{2}x^2 + \frac{1}{2}$



$$\begin{bmatrix} +1 & -1 & & & \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\frac{du}{dx}(0) = 0 \Leftrightarrow \frac{u_1 - u_0}{h} = 0$$

$$u'' = \frac{1}{\Delta x}\left(-u_{i+1} + 2u_i - u_{i-1}\right)$$

$$i = 0 \Rightarrow \quad u'' = \frac{1}{\Delta x}\left(-u_2 + 2u_1 - u_0\right) = \frac{1}{\Delta x}\left(u_1 - u_2\right)$$

First order approximation error on the left hand side

Correction:

$$\frac{du}{dx}(0) = 0 \Leftrightarrow \frac{u_1 - u_{-1}}{\Delta x} = 0$$

$$i = 0 \Rightarrow \quad 1 = \frac{1}{\Delta x}\left(-u_1 + 2u_0 - u_{-1}\right) = \frac{1}{\Delta x}\left(2u_0 - 2u_1\right)$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$