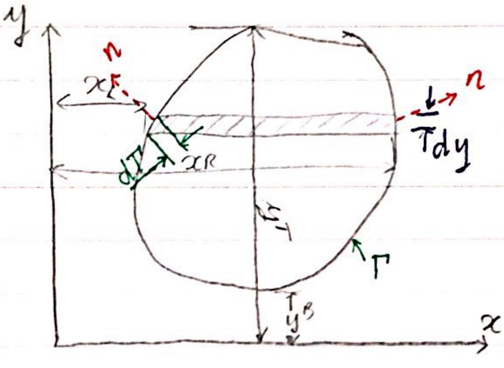
Considering the integration by parts of the following 2D expression:

$$\iint_{\Omega} \phi \frac{\partial \Psi}{\partial x}\, dx\, dy$$

Integrating first with respect to x and using the well-known relation for integration by parts in 1D:

$$\int_{x_L}^{x_R} u\, dv = -\int_{x_L}^{x_R} v\, du + (uv)_{x=x_R} - (uv)_{x=x_L} \quad (1)$$

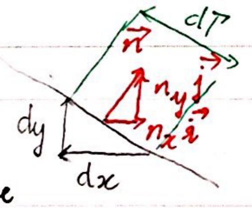
. we have:



$$\iint_{\Omega} \phi \frac{d\Psi}{dx}\, dx\, dy = -\iint_{\Omega} \frac{\partial \phi}{\partial x} \Psi\, dx\, dy + \int_{y_B}^{y^T} \left[(\phi\Psi)_{x=x_R} - (\phi\Psi)_{x=x_L} \right] dy \quad (2)$$

. If now we consider a segment of the boundary $d\Gamma$ on the right hand boundary, we note that:

$$dy = d\Gamma \cdot n_x$$



similarity of 2 triangles: $\frac{dy}{d\Gamma} = \frac{n_x}{1}$

where n_x is the direction cosine between the outward of normal and the x direction.

- Similarly on the left hand size we have:

$$dy = -n_x\, d\Gamma.$$

- The final term of (2) can be expressed as the integral taken along an anticlockwise direction of the complete closed boundary

$$\oint \phi \Psi n_x\, d\Gamma$$

(2) becomes:

$$\iint_{\Omega} \phi \frac{\partial \psi}{\partial x}\, dx\, dy = -\iint_{\Omega} \frac{\partial \phi}{\partial x} \psi\, dx\, dy + \oint_{\Gamma} \phi \psi n_x\, d\Gamma$$

Similarly, if differentiation in the y direction arises we can write

$$\iint_{\Omega} \phi \frac{\partial \psi}{\partial y}\, dx\, dy = -\iint_{\Omega} \frac{\partial \phi}{\partial x} \psi\, dx\, dy + \oint_{\Gamma} \phi \psi n_y\, d\Gamma$$

where n_y is the ~~co~~ direction cosine between the outward normal and the y axis.

* In 3D by identical procedure, we can write:

$$\iiint_{\Omega} \phi \frac{\partial \psi}{\partial y}\, dx\, dy\, dz = -\iiint_{\Omega} \frac{\partial \phi}{\partial y} \psi\, dx\, dy\, dz + \oint_{\Gamma} \phi \psi n_y\, d\Gamma$$

where $d\Gamma$ becomes the elements of the surface area and the last integral is taken over the whole surface.