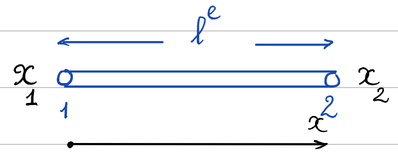
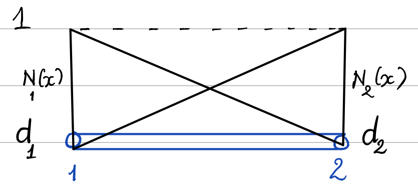
1 Two node linear element:





_ To achieve continuity, we express the approximation in the element in terms of nodal values.

_ To satisfy completeness condition, we need to choose at lease a linear polynominal

$$u^e(x) = a_0^e + a_1^e(x)$$

$$\Leftrightarrow \quad u^e(x) = \underbrace{[1 \;\; x]}_{p(x)} \underbrace{\begin{bmatrix} a_0^e \\ a_1^e \end{bmatrix}}_{a^e} = p(x)\, a^e \qquad (1)$$

_ Now we express the coefficients of a_0^e, a_1^e in term of values of the approximation at nodes 1 and 2:

$$\begin{cases} u^e(x_1) \equiv u_1^e = a_0^e + a_1^e x_1 \\ u^e(x_2) \equiv u_2^e = a_0^e + a_1^e x_2 \end{cases}$$

$$\Leftrightarrow \underbrace{\begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix}}_{d^e} = \underbrace{\begin{bmatrix} 1 & x_1^e \\ 1 & x_2^e \end{bmatrix}}_{M^e} \underbrace{\begin{bmatrix} a_0^e \\ a_2^e \end{bmatrix}}_{a^e} \Rightarrow a^e = (M^e)^{-1} d^e \qquad (2)$$

(2) → (1)

$$u^e(x) = \underbrace{p(x)(M^e)^{-1}}\, d^e$$

$$g^e = [N_1^e(x) \quad N_2^e(x)]^{-1} \text{ : Element shape function matrix}$$

$$\begin{cases} u_1^e(x_1) = N_1^e(x_1)\, d_1 + N_2^e(x_1)\, d_2 \\ u_2^e(x_2) = N_1^e(x_1)\, d_1 + N_2^e(x_2)\, d_2 \end{cases}$$

$$(M^e)^{-1} = \frac{1}{x_2^e - x_1^e} \begin{bmatrix} x_2^e & -x_1^e \\ -1 & 1 \end{bmatrix} = \frac{1}{l^e} \begin{bmatrix} x_2^e & -x_1^e \\ -1 & 1 \end{bmatrix}$$

. $N^e = [N_1^e \;\; N_2^e] = p(x)(M^e)^{-1} = [1 \;\; x] \begin{bmatrix} x_2^e & -x_1^e \\ -1 & 1 \end{bmatrix}$

$$= \frac{1}{l_e}\left[x_2^e - x \qquad -x_1^e + x \right] = \frac{1}{l_e}\left[l^e - x \qquad x \right]$$

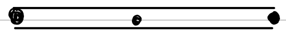
$\Rightarrow N_1^e = (l^e - x)/l^e \quad ; \quad N_2^e = x/l^e$

NOTE:

① $N_I(x_J^e) = \delta_{IJ} = \begin{cases} 1 & \text{if } I = J \\ 0 & \text{if } I \neq J \end{cases}$

② At $x = x_K \Rightarrow u^e(x_K^e) = N_1^e(x_K^e) + N_2^e(x_K^e) = 1$

2 Quadratic 1D element



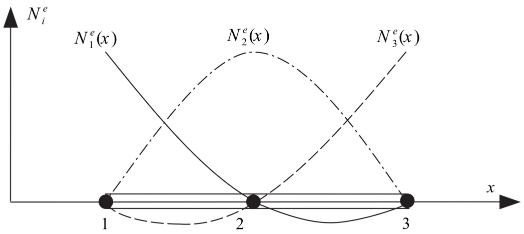


Figure 4.5 The quadratic shape functions for a three-node element.

- We use a complete second-order polynominal approximation

$$u^e(x) = a_0^e + a_1^e x + a_2^e x^2 = \underbrace{\begin{bmatrix} 1 & x & x^2 \end{bmatrix}}_{p(x)} \underbrace{\begin{bmatrix} a_0^e \\ a_1^e \\ a_2^e \end{bmatrix}}_{a^e} \qquad (1)$$

- 2 nodes are placed at the end of the element so that the global approximation will be continuous. The 3^{rd} node is placed at center → symmetrically pleasing.
- We express (a_0^e, a_1^e, a_2^e) in terms of nodal values of:

$$\begin{aligned} u_1^e &= a_0^e + a_1^e x_1^e + a_2^e x_1^e \\ u_2^e &= a_0^e + a_1^e x_2^e + a_2^e x_2^e \\ u_3^e &= a_0^e + a_1^e x_3^e + a_2^e x_3^e \end{aligned} \Rightarrow \underbrace{\begin{bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{bmatrix}}_{d^e} = \underbrace{\begin{bmatrix} 1 & x_1^e & x_1^{e^2} \\ 1 & x_2^e & x_2^{e^2} \\ 1 & x_3^e & x_3^{e^2} \end{bmatrix}}_{M^e} \underbrace{\begin{bmatrix} a_0^e \\ a_e^1 \\ a_e^2 \end{bmatrix}}_{a^e}$$

$$\Rightarrow a^e = (M^e)^{-1} d^e \qquad (2)$$

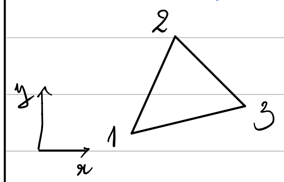
- (1) (2) : $u^e(x) = \underbrace{p(x)(M^e)^{-1}}_{g^e = [N^e(x) \; N_2^e(x) \; N_3^e(x)]} d^e$

$$N_1 = \frac{2}{l_e^2}(x - x_2^e)(x - x_3^e)$$

$$N_2 = -\frac{2}{l_e^2}(x - x_1^e)(x - x_3^e)$$

$$N_3 = -\frac{2}{l_e^2}(x - x_1^e)(x - x_2^e)$$

3) Triangle elements :



$$u^e(x,y) = a_0^e + a_1^e x + a_2^e y$$

$$= \underbrace{[1 \;\; x \;\; y]}_{p(x,y)} \underbrace{\begin{bmatrix} a_0^e \\ a_1^e \\ a_2^e \end{bmatrix}}_{a^e} \qquad (1)$$

- $$\begin{aligned} u_1^e &= a_0^e + a_1^e x_1^e + a_2^e y_1^e \\ u_1^e &= a_0^e + a_1^e x_1^e + a_2^e y_1^e \\ u_1^e &= a_0 + a_1^e x_1^e + a_2^e y_1^e \end{aligned} \Rightarrow \underbrace{\begin{bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{bmatrix}}_{d^e} = \underbrace{\begin{bmatrix} 1 & x_1^e & y_1^e \\ 2 & x_2^e & y_2^e \\ 1 & x_3^e & y_3^e \end{bmatrix}}_{M^e} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}}_{a^e}$$

$$\Rightarrow a^e = (M^e)^{-1} d^e \qquad (2)$$

(2)(1) : $u^e = \underbrace{p(M^e)^{-1}}_{g^e} d^e$

$\hookrightarrow [N_1^e(x,y) \quad N_2^e(x,y) \quad N_3^e(x,y)]$

$$(M^e)^{-1} = \frac{1}{2A^e} \begin{bmatrix} y_2^e - y_3^e & y_3^e - y_1^e & y_1^e - y_2^e \\ x_3^e - x_2^e & x_1^2 - x_3^e & x_2^e - x_1^e \\ x_2^e y_3^e - x_3^e y_2^e & x_3^e y_1^e - x_1^e y_3^e & x_1^e y_2^e - x_2^e y_1^e \end{bmatrix}$$

A^e : area of the element

$$2A^e = \det(M^e) = (x_2^e y_3^e - x_3^e y_2^e) - (x_1^e y_3^e - x_3^e y_1^e) + (x_1^e y_2^e - x_2^e y_1^e)$$

$$\Rightarrow N_1^e = \frac{1}{2A^e}\left\{ x_2^e y_3^e - x_3^e y_2^e + (y_2^e - y_3^e)x + (x_3^e - x_2^e)y \right\}$$

$$N_2^e = \frac{1}{2A^e}\left\{ x_3^e y_1^e - x_1^e y_3^e + (y_3^e - y_1^e)x + (x_1^e - x_3^e)y \right\}$$

$$N_3 = \frac{1}{2A^e}\left\{ x_1^e y_2^e - x_2^e y_1^e + (y_1^e - y_2^e)x + (x_2^e - x_1^e)y \right\}$$

Topic:

Notebook