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Reduced integration:

One obvious approach to avoid the overstiff, locking behavior for incompressible or near-incompressible material is to use reduced integration.

More specifically, since we saw in the previous section that the number of incompressibility constraint in our analysis is equal to the number of quadrature points in the mesh, we can obivously use less quadrature points to reduce the number of constraints.

Uniform reduced integration:

In URI procedure, the number of quadrature point in each direction is one-order lower than that required for full integration

E.g: For the Q4 isoparametric element, a one-point (1×1) quadrature is used instead of the standard, 4-point (2×2) full integration. So, if we use one Gauss point in 2D analysis, we have $\xi_1 = 0$, $\eta_1 = 0$, $W_1 = 4$

The use of URI leads to an undired side effect: rank defficiency of the stiffness matrix, associated with the existences of spurious zero-energy mode.

A 2D Q4 solid element (unrestrained) has 8 dofs. So, the element can exhibit 8 independent displaced configurations. Of these configurations, three correspond to rigid-body motion (2 translation & 1 rotation). Thus a Q4 element must have 5 distinct mode of deformation (i.e, 5 independent displacement configurations must lead to the development of strains & stiffness in the element.) In the context of stiffness matrixes the number of independent displaced configurations to which the element can develope stiffness is called the rank of the stiffness matrix.

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Q4.

Let's examine the case of uniform reduced integr. in a Q4 element.

The stiffness matrix is given by:

$$K^{(e)} = \int_{-1}^{1}\int_{-1}^{1} (B^e)^T D^e B^e J \, d\xi \, d\eta \approx (B_1)^T D_1 B_1 J_1 . 4$$

For a single quadrature point at $\xi = 0$, $\eta = 0$, we can establish an expression for the strain-displacement matrix B_1 at that point:

$$B_1 = \frac{1}{2A^{(e)}} \begin{bmatrix} y_{24}^{(e)} & 0 & y_{31}^{(e)} & 0 & -y_{24}^{(e)} & 0 & -y_{31}^{(e)} & 0 \\ 0 & x_{42}^{(e)} & 0 & x_{13}^{(e)} & 0 & -x_{42}^{(e)} & 0 & -x_{13}^{(e)} \\ x_{42}^{(e)} & y_{24}^{(e)} & x_{13}^{(e)} & y_{31}^{(e)} & -x_{42}^{(e)} & -y_{24}^{(e)} & -x_{13}^{(e)} & -y_{31}^{(e)} \end{bmatrix}$$

where: $x_{ij}^{(e)} = x_i^{(e)} - x_j^{(e)}$; $y_{ij}^{(e)} = y_i^{(e)} - y_j^{(e)}$

and $A^{(e)}$ is the area enclosed by the quadrilateral element:

$$A = \frac{1}{2}\left(x_{42}^{(e)} y_{13}^{(e)} + x_{13}^{(e)} y_{24}^{(e)}\right)$$

The Jacobian J_1 at the single quadrature point can be obtained: $J_1 = A^{(e)}/4$

The rank of a matrix is equal to the number of linearly independent rows or columns that the matrix has (whichever) of the two is smaller. B_1 has 3 rows, which are linearly independent, thus it has a rank equal to 3, and D_1 also has a rank equal to 3.

Note: rank $(A) = m$; rank $(B) = n$; $\Rightarrow$ rank $(A.B) \leq \min(m, n)$

$\Rightarrow$ Rank of $K^{(e)}$ evaluated with one point quadrature is exactly equal to 3.

Thus URI for Q4 element, we have a problem of rank deficiency: the rank of the stiffness matrix is lower than the theoretically require one, which is equal to 5.