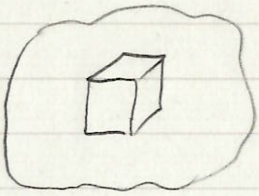
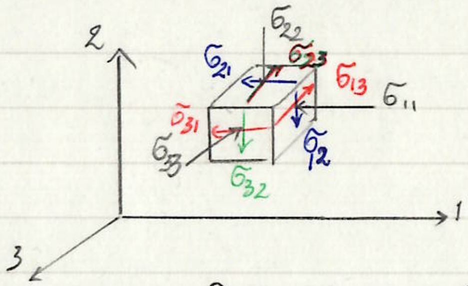
STRESS





- Pick up a point in a continuum body to investigate the stress in this point
- Define the axis
- On each plane, there are 3 stresses: 1 normal & 2 shear

Rule: σ_{ij} i : the name of the plane which the stress acts
j : the direction of the stress

- Put the components of stress in 1 tensor:

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Meaning of stress tensor:

If we want to check stresses on a specific plane, we only need to use the linear transformation of the normal unit vector of that plane by stress tensor

For ex: plane 1 with $\vec{i} = (1, 0, 0)$

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{pmatrix}$$

Conservation of momentum:

$$\sigma_{12} = \sigma_{21} \; ; \; \sigma_{13} = \sigma_{31} \; ; \; \sigma_{23} = \sigma_{32}$$

Mean stress:

The summation of $\sigma_{11}, \sigma_{22}, \sigma_{33}$ has a strong relation with volumetric behavior:

→ Set:

$$p = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3}\sigma_{ii} \quad \text{(Mean stress)}$$

3D Stress Tensor

① Principle stresses - eigen values

General stress tensor can be rotated to find a new stress tensor with principle stresses σ_1, σ_2 and no shear stress.

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \rightarrow \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

Notice that what we have done is that we found diagonal matrix with eigen value in it (tensor with eigen value). Eigen values are in fact the principle stress

② 3D case:

A symmetrix stress tensor: $\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$

If these are real numbers, this tensor must have 3 real eigen values or 3 principle stresses & 3 corresponding eigen vectors which are orthogonal to each other.

③ Find eigen value:

$$M\underset{\text{eigen vector}}{x} = \underset{\text{eigen value}}{\lambda} x \Rightarrow (M - \lambda I)x = 0$$

$$\Rightarrow |M - \lambda I| = 0 \Rightarrow \lambda$$

. For example:

$$\left| \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} - \lambda I \right| = 0 \Leftrightarrow \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0$$

$$\Leftrightarrow (a_{11} - \lambda) \begin{vmatrix} a_{22} - \lambda & a_{23} \\ a_{32} & a_{33} - \lambda \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} - \lambda \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} - \lambda \\ a_{31} & a_{32} \end{vmatrix} = 0$$

$\Rightarrow$ Find λ_1, λ_2, λ_3

④ Find eigen vector:

$$(\sigma - \lambda I)\, x = 0 \rightarrow x_1, x_2, x_3$$

Note: ① $\vec{x}_1 \perp \vec{x}_2$, $\vec{x}_1 \perp \vec{x}_3$, $\vec{x}_2 \perp \vec{x}_3$

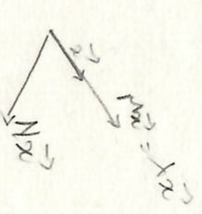
② $\lambda_1 + \lambda_2 + \lambda_3 = \text{trace}(M)$

③ $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = \det(M)$

. Vector $\vec{x}$:

. Tensor M, N

. If:



$\Rightarrow$. $\vec{x}$ is eigen vector of M

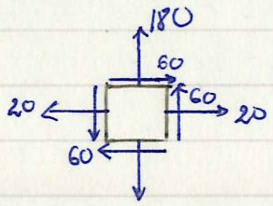
. λ is eigen value of M

STRESS TENSOR

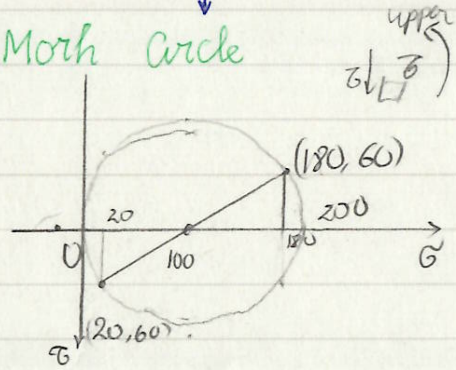
Example:

$$\sigma = \begin{pmatrix} 20 & 60 \\ 60 & 80 \end{pmatrix} \text{MPa}$$ Find

1) Principle stresses?
2) Maximum shear stress



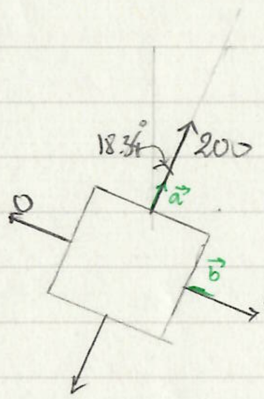
Morh Circle



. Principle stress: 0, 200 MPa

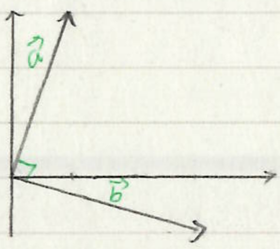
. Maximum shear stress: 100 MPa

. $\tan 2\theta = \frac{60}{80} = \frac{3}{4} \Rightarrow \theta = 18{,}43°$



$\sigma \vec{a}_1 = \lambda_1 \vec{a}_1$

$\sigma \vec{b} = \lambda_2 \vec{b}_1$



Eigen value & Eigen vector

ⓐ Eigen value (principle stress)

$$\sigma - \lambda I = 0$$

$$\Leftrightarrow \begin{vmatrix} 20-\lambda & 60 \\ 60 & 180-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow (20-\lambda)(180-\lambda) - 60^2 = 0$$

$$\Leftrightarrow \left[\begin{array}{l} \lambda_1 = 0 \\ \lambda_2 = 200 \end{array} \right.$$

Note: Characteristic of eigen value of symmetric ~~matrix~~ tensors

① $\lambda_1 + \lambda_2 = \text{trace}(\sigma)$

② $\lambda_1 . \lambda_2 = \det(A)$

ⓑ Eigen vector: Principle direction:

. $\lambda = 0$

$$\Rightarrow \begin{pmatrix} 20 & 60 \\ 60 & 180 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 . \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Leftrightarrow 20x + 60y = 0$$

$$\Leftrightarrow x + 3y = 0$$

$\Rightarrow$ Eigen vector $\vec{a} \begin{pmatrix} 3/\sqrt{10} \\ -1/\sqrt{10} \end{pmatrix}$ (unit vector)

. $\lambda = 200$

$$\Rightarrow \begin{pmatrix} 20 & 60 \\ 60 & 180 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 200 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow 60x = 180y$$

$\Rightarrow$ Eigen vector $\vec{b} = \begin{pmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix}$ (unit vector)

Note: $\vec{a} \perp \vec{b}$ $(a.b = 0)$

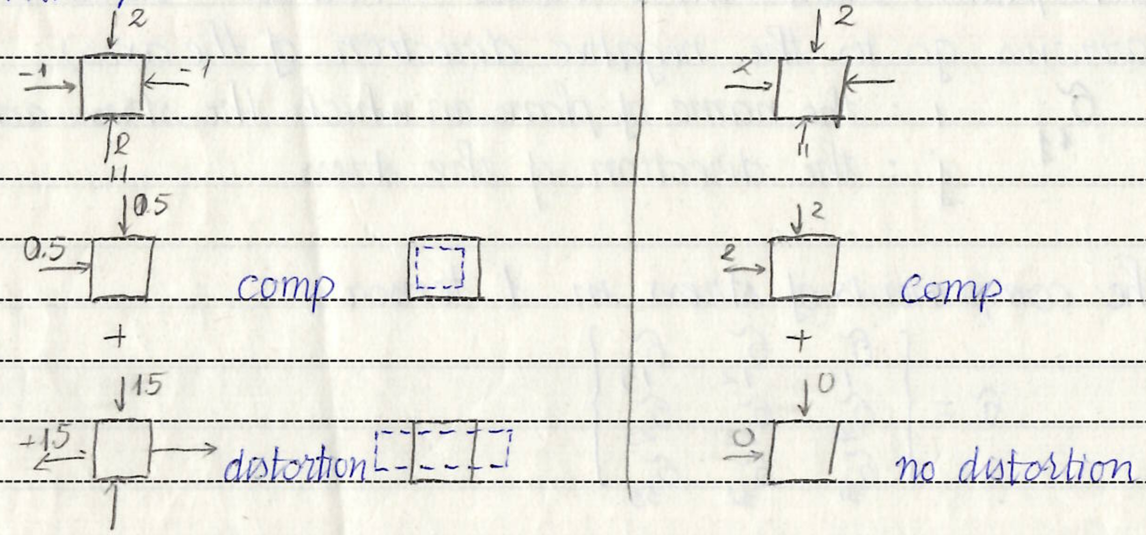
Deviator Stress:

Total stress — Mean stress — Distortional stress tensor

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} - \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} = \begin{pmatrix} \sigma_{11}-p & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-p & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{32}-p \end{pmatrix} = S_{ij}$$ = Deviator Stress tensor

$$\Leftrightarrow \quad \sigma_{ij} - p = S_{ij}$$

Example:



Magnitude of deviator stress tensor:

+ Magnitude of vector: $\|a\| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{a_i a_i}$
+ Magnitude of S_{ij}: $\sqrt{S_{ij} S_{ij}}$

Deviator stress

Definition: $q = \sqrt{\frac{3}{2}} \sqrt{S_{ij} S_{ij}} = \sqrt{\frac{3}{2}} \sqrt{S_{11} + S_{12} + \dots + S_{33}}$

$$q = \sqrt{\frac{3}{2}} \sqrt{(\sigma_{11}-p)^2 + (\sigma_{22}-p)^2 + (\sigma_{33}-p)^2 + 2\sigma_{12}^2 + 2\sigma_{23}^2 + 2\sigma_{13}^2}$$

$$q = \sqrt{\frac{3}{2}} \sqrt{\left(\frac{2\sigma_{11}-\sigma_{22}-\sigma_{33}}{3}\right)^2 + \left(\frac{2\sigma_{22}-\sigma_{11}-\sigma_{33}}{3}\right)^2 + \left(\frac{2\sigma_{33}-\sigma_{11}-\sigma_{22}}{3}\right)^2 + \cancel{2}\sigma_{12}^2 + \cancel{2}\sigma_{23}^2 + \cancel{2}\sigma_{31}^2 + \sigma_{21}^2 + \sigma_{32}^2 + \sigma_{13}^2}$$

Triaxial test: $\sigma_{ij} = 0 \ (i \neq j)$; $\sigma_{22} = \sigma_{33} = \sigma_3$

$$q = \sqrt{\frac{3}{2}} \sqrt{\left(\frac{2\sigma_{11} - 2\sigma_{33}}{3}\right)^2 + 2\left(\frac{\sigma_{33} - \sigma_{11}}{3}\right)^2} = |\sigma_{11} - \sigma_{33}| = \sigma_{11} - \sigma_{33}$$

$$= \sqrt{\frac{3}{2}} \cdot \sqrt{\frac{2}{3}} |\sigma_1 - \sigma_3|$$

STRESS TENSOR

① Determine $\partial p/\partial \sigma_{ij}$:

$$\frac{\partial p}{\partial \sigma_{ij}} = \frac{\partial\left(\frac{1}{3}\sigma_{kk}\right)}{\partial \sigma_{ij}} = \frac{1}{3}\frac{\partial \sigma_{kk}}{\partial \sigma_{ij}} = \frac{1}{3}\delta_{i\underline{k}}\delta_{j\underline{k}} = \frac{1}{3}\delta_{ik}\delta_{kj} = \frac{1}{3}\delta_{ij}$$

② Determine $\partial q/\partial \sigma_{ij}$

$$s_{ij} = \sigma_{ij} - p\delta_{ij} \;;\quad q = \sqrt{\tfrac{3}{2}s_{ij}s_{ij}} \Rightarrow \frac{\partial q}{\partial \sigma_{kl}} = \frac{\partial q}{\partial s_{ij}}\frac{\partial s_{ij}}{\partial \sigma_{ij}}$$

$$\frac{\partial q}{\partial s_{ij}} = \frac{\partial\sqrt{\frac{3}{2}s_{kl}s_{kl}}}{\partial s_{ij}} = \sqrt{\frac{3}{2}}\frac{\partial(s_{kl}.s_{kl})^{1/2}}{\partial s_{ij}}$$

$$= \sqrt{\frac{3}{2}}\cdot\frac{1}{2}\frac{1}{\sqrt{s_{kl}s_{kl}}}\frac{\partial(s_{kl}s_{kl})}{\partial s_{ij}}$$

$$= \sqrt{\frac{3}{2}}\frac{1}{2}\frac{1}{\|s_{kl}\|}2s_{kl} = \sqrt{\frac{3}{2}}\frac{s_{kl}}{\|s_{kl}\|}$$

$$\frac{\partial s_{ij}}{\partial \sigma_{kl}} = \frac{\partial(\sigma_{ij} - p\delta_{ij})}{\partial \sigma_{kl}} = \frac{\partial \sigma_{ij}}{\partial \sigma_{kl}} - \frac{\partial p\delta_{ij}}{\partial \sigma_{kl}}$$

$$= \delta_{ki}\cdot\delta_{lj} - \frac{\partial\left(\frac{1}{3}\right)\sigma_{mm}\delta_{ij}}{\partial \sigma_{kl}}$$

$\delta_{km}\delta_{lm} = \delta_{kl}$

$$= \delta_{ki}\delta_{lj} - \frac{1}{3}\delta_{ij}\frac{\partial \sigma_{mm}}{\partial \sigma_{kl}} = \delta_{ki}\delta_{lj} - \frac{1}{3}\delta_{ij}\delta_{kl}$$

$$\frac{\partial q}{\partial \sigma_{kl}} = \frac{\partial q}{\partial s_{ij}}\frac{\partial s_{ij}}{\partial \sigma_{kl}} = \sqrt{\frac{3}{2}}\frac{s_{kl}}{\|s_{kl}\|}\cdot\left(\delta_{ki}\delta_{lj} - \frac{1}{3}\delta_{ij}\delta_{kl}\right)$$

$$= \sqrt{\frac{3}{2}}\frac{1}{\|s_{kl}\|}\Big(\underbrace{\delta_{ik}s_{kl}\delta_{lj}}_{s_{ij}} - \frac{1}{3}\underbrace{s_{kl}\delta_{lk}}_{s_{kk}\rightarrow 0}\delta_{ij}\Big)$$

$$= \sqrt{\frac{3}{2}}\frac{s_{ij}}{\|s_{mn}\|}$$